

Arithmetical Meadows

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Abstract. An inversive meadow is a commutative ring with identity equipped with a total multiplicative inverse operation satisfying $0^{-1} = 0$. Previously, inversive meadows were shortly called meadows. A divisive meadow is an inversive meadows with the multiplicative inverse operation replaced by a division operation. In the spirit of Peacock's arithmetical algebra, we introduce variants of inversive and divisive meadows without an additive identity element and an additive inverse operation. We give equational axiomatizations of several classes of such variants of inversive and divisive meadows as well as of several instances of them.

Keywords: arithmetical meadow, equational specification, initial algebra specification.

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1 Introduction

The primary mathematical structure for measurement and computation is unquestionably a field. In [8], meadows are proposed as alternatives for fields with a purely equational specification. A meadow is a commutative ring with identity equipped with a total multiplicative inverse operation satisfying two equations which imply that the multiplicative inverse of zero is zero. Thus, meadows are total algebras. As usual in field theory, the convention to consider p / q as an abbreviation for $p \cdot q^{-1}$ was used in subsequent work on meadows (see e.g. [2, 5]). This convention is no longer satisfactory if partial variants of meadows are considered too, as is demonstrated in [3]. In that paper, we rename meadows into inversive meadows and introduce divisive meadows. A divisive meadow is an inversive meadow with the multiplicative inverse operation replaced by the division operation suggested by the above-mentioned abbreviation convention. Henceforth, we will use the name meadow whenever the distinction between inverse meadows and divisive meadows is not important.

Peacock introduced in [15] arithmetical algebra as algebra of numbers where an additive identity element and an additive inverse operation are not involved. That is, arithmetical algebra is algebra of positive numbers instead of algebra of numbers in general (see also [10]). In the spirit of Peacock, we use the name *arithmetical meadow* for a meadow without an additive identity element and an additive inverse operation. We use the name *arithmetical meadow with zero*

for a meadow without an additive inverse operation, but with an additive identity element. Arithmetical meadows related to the field of rational numbers are reminiscent of Peacock's arithmetical algebra.

In this paper, we pursue the following objectives:

1. to complement the signatures of inversive and divisive meadows with arithmetical versions;
2. to provide equational axiomatizations of several classes of arithmetical meadows and instances of them related to the field of rational numbers;
3. to state a number of outstanding questions concerning arithmetical meadows.

This paper is organized as follows. First, we go into the background of the work presented in this paper with the intention to clarify and motivate this work (Section 2). Next, we introduce the classes of inversive and divisive arithmetical meadows and the classes of inversive and divisive arithmetical meadows with zero (Section 3). After that, we introduce instances of these classes related to rational numbers (Section 4). Following this, we have for completeness an interlude on inversive and divisive meadows (Section 5). Then, we state some outstanding questions about arithmetical meadows (Section 6). After that, we shortly discuss partial variants of the instances of the inversive and divisive arithmetical meadows with zero introduced before (Section 7) and an arithmetical version of a well-known mathematical structure closely related to inversive meadows (Section 8). Finally, we make some concluding remarks (Section 9).

2 Background on the Theory of Meadows

In this section, we go into the background of the work presented in this paper with the intention to clarify and motivate this work.

The theory of meadows, see e.g. [2, 5, 3], constitutes a hybrid between the theory of abstract data type and the theory of rings and fields, more specifically the theory of von Neumann regular rings [12, 9] (all fields are von Neumann regular rings).

It is easy to see that each meadow can be reduced to a commutative von Neumann regular ring with a multiplicative identity. Moreover, we know from [2] that each commutative von Neumann regular ring with a multiplicative identity can be expanded to a meadow, and that this expansion is unique. It is easy to show that, if $\phi: X \rightarrow Y$ is an epimorphism between commutative rings with a multiplicative identity and X is a commutative von Neumann regular ring with a multiplicative identity, then: (i) Y is a commutative von Neumann regular ring with a multiplicative identity; (ii) ϕ is also an epimorphism between meadows for the meadows X' and Y' found by means of the unique expansions for X and Y , respectively.

However, there is a difference between commutative von Neumann regular rings with a multiplicative identity and meadows: the class of all meadows is a variety and the class of all commutative von Neumann regular rings with a

multiplicative identity is not. In particular, the class of commutative von Neumann regular rings with a multiplicative identity is not closed under taking subalgebras (a property shared by all varieties). Let \mathcal{Q} be the ring of rational numbers, and let \mathcal{Z} be its subalgebra of integers. Then \mathcal{Q} is a field and for that reason a commutative von Neumann regular ring with a multiplicative identity, but its subalgebra \mathcal{Z} is not a commutative von Neumann regular ring with a multiplicative identity.

In spite of the fact that meadows and commutative von Neumann regular rings with a multiplicative identity are so close that no new mathematics can be expected, there is a difference which matters very much from the perspective of abstract data type specification. \mathcal{Q} , the ring of rational numbers, is not a minimal algebra, whereas \mathcal{Q}_0^i , the inversive meadow of rational numbers is a minimal algebra. As such, \mathcal{Q}_0^i is amenable to initial algebra specification. The first initial algebra specification of \mathcal{Q}_0^i is given in [8] and an improvement due to Hirshfeld is given in [3]. When looking for an initial algebra specification of \mathcal{Q} , adding a total multiplicative inverse operation satisfying $0^{-1} = 0$ as an auxiliary function is the most reasonable solution, assuming that a proper constructor as an auxiliary function is acceptable.

We see a theory of meadows having two roles: (i) a starting-point of a theory of mathematical data types; (ii) an intermediate between algebra and logic.

On investigation of mathematical data types, known countable mathematical structures will be equipped with operations to obtain minimal algebras and specification properties of those minimal algebras will be investigated. If countable minimal algebras can be classified as either computable, semi-computable or co-semi-computable, known specification techniques may be applied (see [7] for a survey of this matter). Otherwise data type specification in its original forms cannot be applied. Further, one may study ω -completeness of specifications and term rewriting system related properties.

It is not a common viewpoint in algebra or in mathematics at large that giving a name to an operation, which is included in a signature, is a very significant step by itself. However, the answer to the notorious question “what is $1 / 0$ ” is very sensitive to exactly this matter. Von Neumann regular rings provide a classical mathematical perspective on rings and fields, where multiplicative inverse (or division) is only used when its use is clearly justified and puzzling uses are rejected as a matter of principle. Meadows provide a more logical perspective to von Neumann regular rings in which justified and unjustified use of multiplicative inverse cannot be easily distinguished beforehand.

Arithmetical meadows, in particular the ones related to the field of rational numbers, provide additional insight in what is yielded by the presence of an operator for multiplicative inverse (or division) in a signature.

3 Equational Specifications of Arithmetical Meadows

This section concerns the equational specification of several classes of arithmetical meadows.

The signature of commutative rings with a multiplicative identity consists of the following constants and operators:

- the *additive identity* constant 0;
- the *multiplicative identity* constant 1;
- the binary *addition* operator $+$;
- the binary *multiplication* operator \cdot ;
- the unary *additive inverse* operator $-$;

The signature of inversive meadows consists of the constants and operators from the signature of commutative rings with a multiplicative identity and in addition:

- the unary *multiplicative inverse* operator $^{-1}$.

The signature of divisive meadows consists of the constants and operators from the signature of commutative rings with a multiplicative identity and in addition:

- the binary *division* operator $/$.

The signatures of inversive and divisive arithmetical meadows with zero are the signatures of inversive and divisive meadows with the additive inverse operator $-$ removed. The signatures of inversive and divisive arithmetical meadows are the signatures of inversive and divisive arithmetical meadows with zero with the additive identity constant 0 removed. We write:

$$\begin{aligned}
\Sigma_{\text{CR}} & \text{ for } \{0, 1, +, \cdot, -\}, \\
\Sigma_{\text{Md}}^{\text{i}} & \text{ for } \Sigma_{\text{CR}} \cup \{-^1\}, \\
\Sigma_{\text{Md}}^{\text{d}} & \text{ for } \Sigma_{\text{CR}} \cup \{/\}, \\
\Sigma_{\text{AMd}}^{\text{iz}} & \text{ for } \Sigma_{\text{Md}}^{\text{i}} \setminus \{-\}, \\
\Sigma_{\text{AMd}}^{\text{dz}} & \text{ for } \Sigma_{\text{Md}}^{\text{d}} \setminus \{-\}, \\
\Sigma_{\text{AMd}}^{\text{i}} & \text{ for } \Sigma_{\text{AMd}}^{\text{iz}} \setminus \{0\}, \\
\Sigma_{\text{AMd}}^{\text{d}} & \text{ for } \Sigma_{\text{AMd}}^{\text{dz}} \setminus \{0\}.
\end{aligned}$$

We use infix notation for the binary operators, prefix notation for the unary operator $-$, and postfix notation for the unary operator $^{-1}$. We use the usual precedence convention to reduce the need for parentheses. We denote the numerals 0, 1, $1 + 1$, $(1 + 1) + 1$, ... by $\underline{0}$, $\underline{1}$, $\underline{2}$, $\underline{3}$, ... and we use the notation p^n for exponentiation with a natural number as exponent. Formally, we define \underline{n} inductively by $\underline{0} = 0$, $\underline{1} = 1$ and $\underline{n+2} = \underline{n} + 1$ and we define, for each term p over the signature of inversive meadows or the signature of divisive meadows, p^n inductively by $p^0 = 1$ and $p^{n+1} = p^n \cdot p$.

The axioms of a commutative ring with a multiplicative identity are the equations given in Table 1. We write:

$$\begin{aligned}
E_{\text{CR}} & \text{ for the set of all equations in Table 1,} \\
E_{\text{CR}_{\text{az}}} & \text{ for } E_{\text{CR}} \setminus \{x + (-x) = 0\}, \\
E_{\text{CR}_{\text{a}}} & \text{ for } E_{\text{CR}_{\text{az}}} \setminus \{x + 0 = x\}.
\end{aligned}$$

Table 1. Axioms of a commutative ring with a multiplicative identity

$(x + y) + z = x + (y + z)$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x + y = y + x$	$x \cdot y = y \cdot x$
$x + 0 = x$	$x \cdot 1 = x$
$x + (-x) = 0$	$x \cdot (y + z) = x \cdot y + x \cdot z$

The equations in $E_{\text{CR}_{\text{az}}}$ are the equations from E_{CR} in which the additive inverse operator $-$ does not occur. The equations in E_{CR_a} are the equations from $E_{\text{CR}_{\text{az}}}$ in which the additive identity constant 0 does not occur.

To axiomatize inversive and divisive arithmetical meadows, we need additional equations. We write:

$$E_{\text{AMd}}^i \text{ for } E_{\text{CR}_a} \cup \{x \cdot x^{-1} = 1\},$$

$$E_{\text{AMd}}^d \text{ for } E_{\text{CR}_a} \cup \{x / x = 1\}.$$

The class of *inversive arithmetical meadows* is the class of all algebras over the signature Σ_{AMd}^i that satisfy the equations E_{AMd}^i and the class of *divisive arithmetical meadows* is the class of all algebras over the signature Σ_{AMd}^d that satisfy the equations E_{AMd}^d .

We state and prove two equational facts about numerals that will be used in later proofs.

Lemma 1. *For all $n, m \in \mathbb{N} \setminus \{0\}$, we have that $E_{\text{CR}_a} \vdash \underline{n + m} = \underline{n} + \underline{m}$ and $E_{\text{CR}_a} \vdash \underline{n \cdot m} = \underline{n} \cdot \underline{m}$.*

Proof. The fact that $\underline{n + m} = \underline{n} + \underline{m}$ is derivable from E_{CR_a} is easily proved by induction on n . The basis step is trivial. The inductive step goes as follows: $\underline{(n + 1) + m} = \underline{(n + m) + 1} = \underline{n + m} + 1 = \underline{n} + \underline{m} + 1 = \underline{n} + 1 + \underline{m} = \underline{n + 1} + \underline{m}$. The fact that $\underline{n \cdot m} = \underline{n} \cdot \underline{m}$ is derivable from E_{CR_a} is easily proved by induction on n , using that $\underline{n + m} = \underline{n} + \underline{m}$ is derivable from E_{CR_a} . The basis step is trivial. The inductive step goes as follows: $\underline{(n + 1) \cdot m} = \underline{n \cdot m + 1 \cdot m} = \underline{n \cdot m} + \underline{1 \cdot m} = \underline{n \cdot m} + \underline{1} \cdot \underline{m} = \underline{n} \cdot \underline{m} + \underline{1} \cdot \underline{m} = (\underline{n} + \underline{1}) \cdot \underline{m} = \underline{n + 1} \cdot \underline{m}$. \square

We state and prove two useful facts about the multiplicative inverse operator derivable from E_{AMd}^i .

Lemma 2. *We have $E_{\text{AMd}}^i \vdash (x^{-1})^{-1} = x$ and $E_{\text{AMd}}^i \vdash (x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$.*

Proof. We derive $(x^{-1})^{-1} = x$ from E_{AMd}^i as follows: $(x^{-1})^{-1} = 1 \cdot (x^{-1})^{-1} = (x \cdot x^{-1}) \cdot (x^{-1})^{-1} = x \cdot (x^{-1} \cdot (x^{-1})^{-1}) = x \cdot 1 = x$. We derive $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$ from E_{AMd}^i as follows: $(x \cdot y)^{-1} = 1 \cdot (1 \cdot (x \cdot y)^{-1}) = (x \cdot x^{-1}) \cdot ((y \cdot y^{-1}) \cdot (x \cdot y)^{-1}) = (x^{-1} \cdot y^{-1}) \cdot ((x \cdot y) \cdot (x \cdot y)^{-1}) = (x^{-1} \cdot y^{-1}) \cdot 1 = x^{-1} \cdot y^{-1}$. \square

To axiomatize inversive and arithmetical meadows with zero, we need other additional equations. We write:

$$E_{\text{AMd}}^{\text{iz}} \text{ for } E_{\text{CR}_{\text{az}}} \cup \{(x^{-1})^{-1} = x, x \cdot (x \cdot x^{-1}) = x\},$$

$$E_{\text{AMd}}^{\text{dz}} \text{ for } E_{\text{CR}_{\text{az}}} \cup \{1 / (1 / x) = x, (x \cdot x) / x = x, x / y = x \cdot (1 / y)\}.$$

The class of *inversive arithmetical meadows with zero* is the class of all algebras over the signature $\Sigma_{\text{AMd}}^{\text{iz}}$ that satisfy the equations $E_{\text{AMd}}^{\text{iz}}$ and the class of *divisive arithmetical meadows with zero* is the class of all algebras over the signature $\Sigma_{\text{AMd}}^{\text{dz}}$ that satisfy the equations $E_{\text{AMd}}^{\text{dz}}$.

We state and prove two useful facts about the additive identity constant derivable from $E_{\text{AMd}}^{\text{iz}}$.

Lemma 3. *We have $E_{\text{AMd}}^{\text{iz}} \vdash 0 \cdot x = 0$ and $E_{\text{AMd}}^{\text{iz}} \vdash 0^{-1} = 0$.*

Proof. Firstly, we derive $x + y = x \Rightarrow y = 0$ from $E_{\text{AMd}}^{\text{iz}}$ as follows: $x + y = x \Rightarrow 0 + y = 0 \Rightarrow y + 0 = 0 \Rightarrow y = 0$. Secondly, we derive $x + 0 \cdot x = x$ from $E_{\text{AMd}}^{\text{iz}}$ as follows: $x + 0 \cdot x = x \cdot 1 + 0 \cdot x = 1 \cdot x + 0 \cdot x = (1 + 0) \cdot x = 1 \cdot x = x \cdot 1 = x$. From $x + y = x \Rightarrow y = 0$ and $x + 0 \cdot x = x$, it follows that $0 \cdot x = 0$. We derive $0^{-1} = 0$ from $E_{\text{AMd}}^{\text{iz}}$ as follows: $0^{-1} = 0^{-1} \cdot (0^{-1} \cdot (0^{-1})^{-1}) = (0^{-1})^{-1} \cdot (0^{-1} \cdot 0^{-1}) = 0 \cdot (0^{-1} \cdot 0^{-1}) = 0$. \square

We state and prove a useful fact about the multiplicative inverse operator derivable from $E_{\text{AMd}}^{\text{iz}}$.

Lemma 4. *We have $E_{\text{AMd}}^{\text{iz}} \vdash (x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$.*

Proof. Proposition 2.8 from [2] states that $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$ is derivable from $E_{\text{AMd}}^{\text{iz}} \cup \{x + 0 = x, x + (-x) = 0\}$. The proof of this proposition given in [2] goes through because no use is made of the equations $x + 0 = x$ and $x + (-x) = 0$. \square

We state and prove a fact about the additive identity constant that will be used in a later proof.

Lemma 5. *For each $\Sigma_{\text{AMd}}^{\text{iz}}$ -term t , either $E_{\text{AMd}}^{\text{iz}} \vdash t = 0$ or there exists a $\Sigma_{\text{AMd}}^{\text{i}}$ -term t' such that $E_{\text{AMd}}^{\text{iz}} \vdash t = t'$.*

Proof. The proof is easy by induction on the structure of t , using Lemma 3. \square

4 Arithmetical Meadows of Rational Numbers

We obtain inverse and divisive arithmetical meadows closely related to the field of rational numbers as the initial algebras of equational specifications. As usual, we write $I(\Sigma, E)$ for the initial algebra among the algebras over the signature Σ that satisfy the equations E (see e.g. [6]).

\mathcal{Q}^{ia} , the inversive arithmetical meadow of rational numbers, is defined as follows:

$$\mathcal{Q}^{\text{ia}} = I(\Sigma_{\text{AMd}}^{\text{i}}, E_{\text{AMd}}^{\text{i}}).$$

\mathcal{Q}^{da} , the divisive arithmetical meadow of rational numbers, is defined as follows:

$$\mathcal{Q}^{\text{da}} = I(\Sigma_{\text{AMd}}^{\text{d}}, E_{\text{AMd}}^{\text{d}}).$$

\mathcal{Q}^{ia} and \mathcal{Q}^{da} are the initial algebras in the class of inversive arithmetical meadows and the class of divisive arithmetical meadows, respectively.

Theorem 1. $\mathcal{Q}^{\text{ia}} = I(\Sigma_{\text{AMd}}^{\text{i}}, E_{\text{AMd}}^{\text{i}})$ is the subalgebra of the $\Sigma_{\text{AMd}}^{\text{i}}$ -reduct of the inversive meadow of rational numbers whose domain is the set of all positive rational numbers.

Proof. Like in the case of Theorem 3.1 from [8], it is sufficient to prove that, for each closed term t over the signature $\Sigma_{\text{AMd}}^{\text{i}}$, there exists a unique term t' in the set

$$\{\underline{n} \cdot \underline{m}^{-1} \mid n, m \in \mathbb{N} \setminus \{0\} \wedge \gcd(n, m) = 1\}$$

such that $E_{\text{AMd}}^{\text{i}} \vdash t = t'$. Like in the case of Theorem 3.1 from [8], this is proved by induction on the structure of t , using Lemmas 1 and 2. The proof is similar, but simpler owing to: (i) the absence of terms of the forms 0 and $-t'$; (ii) the absence of terms of the forms $\underline{0}$ and $-(\underline{n} \cdot \underline{m}^{-1})$ among the terms that exist by the induction hypothesis; (iii) the presence of the axiom $x \cdot x^{-1} = 1$. \square

The fact that \mathcal{Q}^{da} is the initial algebra in the class of divisive arithmetical meadows is proved similarly.

Derivability of equations from the equations of the initial algebra specifications of \mathcal{Q}^{ia} and \mathcal{Q}^{da} is decidable.

Theorem 2. For all $\Sigma_{\text{AMd}}^{\text{i}}$ -terms t and t' , it is decidable whether $E_{\text{AMd}}^{\text{i}} \vdash t = t'$.

Proof. For each $\Sigma_{\text{AMd}}^{\text{i}}$ -term r , there exist $\Sigma_{\text{AMd}}^{\text{i}}$ -terms r_1 and r_2 in which the multiplicative inverse operator does not occur such that $E_{\text{AMd}}^{\text{i}} \vdash r = r_1 \cdot r_2^{-1}$. The proof of this fact is easy by induction on the structure of r , using Lemma 2. Inspection of the proof yields that there is an effective way to find witnessing terms.

For each closed $\Sigma_{\text{AMd}}^{\text{i}}$ -term r in which the multiplicative inverse operator does not occur there exists a $k \in \mathbb{N} \setminus \{0\}$, such that $E_{\text{AMd}}^{\text{i}} \vdash r = \underline{k}$. The proof of this fact is easy by induction on the structure of r . Moreover, for each $\Sigma_{\text{AMd}}^{\text{i}}$ -term r in which the multiplicative inverse operator does not occur there exists a $\Sigma_{\text{AMd}}^{\text{i}}$ -term r' of the form $\sum_{i_1=1}^{n_1} \dots \sum_{i_m=1}^{n_m} k_{i_1 \dots i_m} \cdot x_1^{i_1} \cdot \dots \cdot x_m^{i_m}$, where $k_{i_1 \dots i_m} \in \mathbb{N} \setminus \{0\}$ for each $i_1 \in [1, n_1], \dots, i_m \in [1, n_m]$ and x_1, \dots, x_m are variables, such that $E_{\text{AMd}}^{\text{i}} \vdash r = r'$. The proof of this fact is easy by induction on the structure of r , using the previous fact. Inspection of the proof yields that there is an effective way to find a witnessing term. Terms of the form described above are polynomials in several variables with positive integer coefficients.

Let t_1, t_2, t'_1, t'_2 be $\Sigma_{\text{AMd}}^{\text{i}}$ -terms in which the multiplicative inverse operator does not occur such that $E_{\text{AMd}}^{\text{i}} \vdash t = t_1 \cdot t_2^{-1}$ and $E_{\text{AMd}}^{\text{i}} \vdash t' = t'_1 \cdot t'_2^{-1}$. Moreover, let s and s' be $\Sigma_{\text{AMd}}^{\text{i}}$ -terms of the form $\sum_{i_1=1}^{n_1} \dots \sum_{i_m=1}^{n_m} k_{i_1 \dots i_m} \cdot x_1^{i_1} \cdot \dots \cdot x_m^{i_m}$, where $k_{i_1 \dots i_m} \in \mathbb{N} \setminus \{0\}$ for each $i_1 \in [1, n_1], \dots, i_m \in [1, n_m]$ and x_1, \dots, x_m are variables, such that $E_{\text{AMd}}^{\text{i}} \vdash t_1 \cdot t'_2 = s$ and $E_{\text{AMd}}^{\text{i}} \vdash t'_1 \cdot t_2 = s'$. We have that $E_{\text{AMd}}^{\text{i}} \vdash t = t'$ iff $E_{\text{AMd}}^{\text{i}} \vdash t_1 \cdot t_2^{-1} = t'_1 \cdot t'_2^{-1}$ iff $E_{\text{AMd}}^{\text{i}} \vdash t_1 \cdot t'_2 = t'_1 \cdot t_2$ iff $E_{\text{AMd}}^{\text{i}} \vdash s = s'$. Moreover, we have that $E_{\text{AMd}}^{\text{i}} \vdash s = s'$ only if s and s' denote the same function on positive real numbers in the inversive arithmetical meadow of positive real numbers. The latter is decidable because polynomials in several variables with positive integer coefficients denote the same function on positive

real numbers in the inversive arithmetical meadow of positive real numbers only if they are syntactically equal. \square

The fact that derivability of equations from the equations of the initial algebra specification of \mathcal{Q}^{da} is decidable is proved similarly.

We obtain inverse and divisive arithmetical meadows with zero closely related to the field of rational numbers as the initial algebras of equational specifications.

$\mathcal{Q}_0^{\text{iaz}}$, the inversive arithmetical meadow of rational numbers with zero, is defined as follows:

$$\mathcal{Q}_0^{\text{iaz}} = I(\Sigma_{\text{AMd}}^{\text{iz}}, E_{\text{AMd}}^{\text{iz}} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\}).$$

$\mathcal{Q}_0^{\text{daz}}$, the divisive arithmetical meadow of rational numbers with zero, is defined as follows:

$$\mathcal{Q}_0^{\text{daz}} = I(\Sigma_{\text{AMd}}^{\text{dz}}, E_{\text{AMd}}^{\text{dz}} \cup \{(1 + x^2 + y^2) / (1 + x^2 + y^2) = 1\}).$$

$\mathcal{Q}_0^{\text{iaz}}$ and $\mathcal{Q}_0^{\text{daz}}$ are the initial algebras in the class of inversive arithmetical meadows with zero that satisfy $(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1$ and the class of divisive arithmetical meadows with zero that satisfy $(1 + x^2 + y^2) / (1 + x^2 + y^2) = 1$, respectively. First we prove a fact that is useful in proving that $\mathcal{Q}_0^{\text{iaz}}$ is the initial algebra in the class of inversive arithmetical meadows with zero.

Lemma 6. *In $I(\Sigma_{\text{AMd}}^{\text{iz}}, E_{\text{AMd}}^{\text{iz}} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\})$, \underline{n} has a multiplicative inverse for each $n \in \mathbb{N}$.*

Proof. In the proof of Theorem 3 from [3] given in that paper, it is among other things proved that \underline{n} has a multiplicative inverse for each $n \in \mathbb{N}$ in $I(\Sigma_{\text{AMd}}^{\text{i}}, E_{\text{AMd}}^{\text{iz}} \cup \{x + 0 = x, x + (-x) = 0, \} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\})$. The proof concerned goes through because no use is made of the equations $x + 0 = x$ and $x + (-x) = 0$. \square

Theorem 3. *$\mathcal{Q}_0^{\text{iaz}} = I(\Sigma_{\text{AMd}}^{\text{iz}}, E_{\text{AMd}}^{\text{iz}} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\})$ is the subalgebra of the $\Sigma_{\text{AMd}}^{\text{iz}}$ -reduct of the inversive meadow of rational numbers whose domain is the set of all non-negative rational numbers.*

Proof. Like in the case of Theorem 1, it is sufficient to prove that, for each closed term t over the signature $\Sigma_{\text{AMd}}^{\text{i}}$, there exists a unique term t' in the set

$$\{\underline{0}\} \cup \{\underline{n} \cdot \underline{m}^{-1} \mid n, m \in \mathbb{N} \setminus \{0\} \wedge \gcd(n, m) = 1\}$$

such that $E_{\text{AMd}}^{\text{iz}} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\} \vdash t = t'$. Like in the case of Theorem 1, this is proved by induction on the structure of t , now using Lemmas 1, 3 and 4. The proof is similar, but more complicated owing to: (i) the presence of terms of the form $\underline{0}$; (ii) the presence of terms of the form $\underline{0}$ among the terms that exist by the induction hypothesis; (iii) the absence of the axiom $x \cdot x^{-1} = 1$. Because of the last point, use is made of Lemma 6. \square

The fact that $\mathcal{Q}_0^{\text{daz}}$ is the initial algebra in the class of divisive arithmetical meadows with zero is proved similarly.

An alternative initial algebra specification of $\mathcal{Q}_0^{\text{iaz}}$ is obtained if the equation $(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1$ is replaced by $(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}$.

Theorem 4. $\mathcal{Q}_0^{\text{iaz}} \cong I(\Sigma_{\text{AMd}}^{\text{iz}}, E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\})$.

Proof. It is sufficient to prove that $(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}$ is valid in $\mathcal{Q}_0^{\text{iaz}}$ and $(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1$ is valid in $I(\Sigma_{\text{AMd}}^{\text{iz}}, E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\})$. It follows from Lemma 4, and the associativity and commutativity of \cdot , that $(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1} \Leftrightarrow (x \cdot x^{-1}) \cdot ((x+y) \cdot (x+y)^{-1}) = x \cdot x^{-1}$ is derivable from $E_{\text{AMd}}^{\text{iz}}$. This implies that $(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}$ is valid in $\mathcal{Q}_0^{\text{iaz}}$ iff $(x \cdot x^{-1}) \cdot ((x+y) \cdot (x+y)^{-1}) = x \cdot x^{-1}$ is valid in $\mathcal{Q}_0^{\text{iaz}}$. The latter is easily established by distinction between the cases $x = 0$ and $x \neq 0$. To show that $(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1$ is valid in $I(\Sigma_{\text{AMd}}^{\text{iz}}, E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\})$, it is sufficient to derive $(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1$ from $E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot x^{-1}) \cdot ((x+y) \cdot (x+y)^{-1}) = x \cdot x^{-1}\}$. The derivation is fully trivial with the exception of the first step, viz. substituting 1 for x and $x^2 + y^2$ for y in $(x \cdot x^{-1}) \cdot ((x+y) \cdot (x+y)^{-1}) = x \cdot x^{-1}$. \square

An alternative initial algebra specification of $\mathcal{Q}_0^{\text{daz}}$ is obtained in the same vein.

In $\mathcal{Q}_0^{\text{iaz}}$, the *general inverse law* $x \neq 0 \Rightarrow x \cdot x^{-1} = 1$ is valid. Derivability of equations from the equations of the alternative initial algebra specification of $\mathcal{Q}_0^{\text{iaz}}$ and the general inverse law is decidable. First we prove a fact that is useful in proving this decidability result.

Lemma 7. For all $\Sigma_{\text{AMd}}^{\text{i}}$ -terms t in which no other variables than x_1, \dots, x_n occur, $E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\} \cup \{x_1 \cdot x_1^{-1} = 1, \dots, x_n \cdot x_n^{-1} = 1\} \vdash_{x_1, \dots, x_n} t \cdot t^{-1} = 1$.

Proof. The proof is easy by induction on the structure of t , using Lemma 4. \square

Theorem 5. For all $\Sigma_{\text{AMd}}^{\text{iz}}$ -terms t and t' , it is decidable whether $E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\} \cup \{x \neq 0 \Rightarrow x \cdot x^{-1} = 1\} \vdash t = t'$.

Proof. Let $E_{\text{AMd}}^{\text{iz}+} = E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\} \cup \{x \neq 0 \Rightarrow x \cdot x^{-1} = 1\}$. We prove that $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ is decidable by induction on the number of variables occurring in $t = t'$. In the case where the number of variables is 0, we have that $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ iff $\mathcal{Q}_0^{\text{iaz}} \models t = t'$ iff $E_{\text{AMd}}^{\text{iz}} \cup \{(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1\} \vdash t = t'$. The last is decidable because, by the proof of Theorem 3, there exist unique terms s and s' in the set $\{\underline{0}\} \cup \{\underline{n} \cdot \underline{m}^{-1} \mid n, m \in \mathbb{N} \setminus \{0\} \wedge \gcd(n, m) = 1\}$ such that $E_{\text{AMd}}^{\text{iz}} \cup \{(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1\} \vdash t = s$ and $E_{\text{AMd}}^{\text{iz}} \cup \{(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1\} \vdash t' = s'$, and inspection of that proof yields that there is an effective way to find s and s' . Hence, in the case where the number of variables is 0, $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ is decidable. In the case where the number of variables is $n+1$, suppose that the variables are x_1, \dots, x_{n+1} . Let s be such that $E_{\text{AMd}}^{\text{iz}} \vdash t = s$ and s is either a $\Sigma_{\text{AMd}}^{\text{i}}$ -term or the constant 0 and let s' be such that $E_{\text{AMd}}^{\text{iz}} \vdash t' = s'$ and s' is either a $\Sigma_{\text{AMd}}^{\text{i}}$ -term or the constant 0. Such s and s' exist by Lemma 5, and inspection of the proof of that lemma yields that there is

an effective way to find s and s' . We have that $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ iff $E_{\text{AMd}}^{\text{iz}+} \vdash s = s'$. In the case where not both s and s' are Σ_{AMd}^i -terms, $E_{\text{AMd}}^{\text{iz}+} \vdash s = s'$ only if s and s' are syntactically equal. Hence, in this case, $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ is decidable. In the case where both s and s' are Σ_{AMd}^i -terms, by the general inverse law, we have that $E_{\text{AMd}}^{\text{iz}+} \vdash s = s'$ iff $E_{\text{AMd}}^{\text{iz}+} \vdash s[0/x_i] = s'[0/x_i]$ for all $i \in [1, n+1]$ and $E_{\text{AMd}}^{\text{iz}+} \cup \{x_1 \cdot x_1^{-1} = 1, \dots, x_{n+1} \cdot x_{n+1}^{-1} = 1\} \vdash_{x_1, \dots, x_{n+1}} s = s'$. By Lemma 7, we have that $E_{\text{AMd}}^{\text{iz}+} \cup \{x_1 \cdot x_1^{-1} = 1, \dots, x_{n+1} \cdot x_{n+1}^{-1} = 1\} \vdash_{x_1, \dots, x_{n+1}} s = s'$ iff $E_{\text{AMd}}^i \vdash s = s'$. For each $i \in [1, n+1]$, $E_{\text{AMd}}^{\text{iz}+} \vdash s[0/x_i] = s'[0/x_i]$ is decidable because the number of variables occurring in $s[0/x_i] = s'[0/x_i]$ is n . Moreover, we know from Theorem 2 that $E_{\text{AMd}}^i \vdash s = s'$ is decidable. Hence, in the case where both s and s' are Σ_{AMd}^i -terms, $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ is decidable as well. \square

The fact that derivability of equations from the equations of the alternative initial algebra specification of $\mathcal{Q}_0^{\text{daz}}$ and $x \neq 0 \Rightarrow x / x = 1$ is decidable is proved similarly. We remark that it is an open problem whether derivability of equations from the equations of the alternative initial algebra specifications of $\mathcal{Q}_0^{\text{iaz}}$ and $\mathcal{Q}_0^{\text{daz}}$ is decidable.

5 Interlude on Meadows

For completeness, we shortly discuss inversive and divisive meadows.

To axiomatize inversive and divisive meadows, we need the following sets of equations:

$$\begin{aligned} E_{\text{Md}}^i &\text{ for } E_{\text{CR}} \cup \{(x^{-1})^{-1} = x, x \cdot (x \cdot x^{-1}) = x\}, \\ E_{\text{Md}}^d &\text{ for } E_{\text{CR}} \cup \{1 / (1 / x) = x, (x \cdot x) / x = x, x / y = x \cdot (1 / y)\}. \end{aligned}$$

The class of *inversive meadows* is the class of all algebras over the signature Σ_{Md}^i that satisfy the equations E_{Md}^i ; and the class of *divisive meadows* is the class of all algebras over the signature Σ_{Md}^d that satisfy the equations E_{Md}^d .

A meadow is *non-trivial* if it satisfies $0 \neq 1$; and a meadow is a *cancellation meadow* if it satisfies $x \neq 0 \wedge x \cdot y = x \cdot z \Rightarrow y = z$. In [8], an inversive cancellation meadow is called a *zero-totalized field*.

Recently, we found out in [14] that inversive meadows were already introduced by Komori [11] in a report from 1975, where they go by the name of *desirable pseudo-fields*. We propose the name *Komori ring* as an alternative for inversive meadow and the name *Komori field* as an alternative for inversive non-trivial cancellation meadow. In [14], we also found an axiomatization of inversive meadows which differs from the one given above. We came across this paper by a reference in [16].

\mathcal{Q}_0^i , the inversive meadow of rational numbers, is defined as follows:

$$\mathcal{Q}_0^i = I(\Sigma_{\text{Md}}^i, E_{\text{Md}}^i \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\}).$$

\mathcal{Q}_0^d , the divisive meadow of rational numbers, is defined as follows:

$$\mathcal{Q}_0^d = I(\Sigma_{\text{Md}}^d, E_{\text{Md}}^d \cup \{(1 + x^2 + y^2) / (1 + x^2 + y^2) = 1\}).$$

Moss showed in [13] that there exists an initial algebra specification of \mathcal{Q} with just one hidden function. The initial algebra specification of \mathcal{Q}_0^i given above is without hidden functions.

Inversive meadows have been extended with signum, floor and ceiling operations in [5], differentiation operations in [4], and a square root operation in [1].

6 Outstanding Questions about Arithmetical Meadows

The following are some outstanding questions with regard to arithmetical meadows:

1. Is the initial algebra specification of \mathcal{Q}_0^i a conservative extension of the initial algebra specifications of \mathcal{Q}^{ia} and $\mathcal{Q}_0^{\text{iaz}}$?
2. Do \mathcal{Q}^{ia} and $\mathcal{Q}_0^{\text{iaz}}$ have initial algebra specifications that constitute complete term rewriting systems (modulo associativity and commutativity of $+$ and \cdot)?
3. Do \mathcal{Q}^{ia} and $\mathcal{Q}_0^{\text{iaz}}$ have ω -complete initial algebra specifications?
4. What are the complexities of derivability of equations from E_{AMd}^i and $E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x + y)) \cdot (x \cdot (x + y))^{-1} = x \cdot x^{-1}, x \neq 0 \Rightarrow x \cdot x^{-1} = 1\}$?
5. Is derivability of equations from $E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x + y)) \cdot (x \cdot (x + y))^{-1} = x \cdot x^{-1}\} \vdash t = t'$ decidable?
6. Do we have $\mathcal{Q}_0^{\text{iaz}} \cong I(\Sigma_{\text{AMd}}^{\text{iz}}, E_{\text{AMd}}^{\text{iz}} \cup \{(1 + x^2) \cdot (1 + x^2)^{-1} = 1\})$?

These questions are formulated for the inversive case, but they have counterparts for the divisive case of which some might lead to different answers.

7 Partial Arithmetical Meadows with Zero

Following [3], we introduce in this section simple constructions of partial inversive and divisive arithmetical meadows with zero from total ones.

We take the position that partial algebras should be made from total ones. For the case that we are engaged in, this means that relevant partial arithmetical meadows with zero are obtained from arithmetical meadows with zero by making certain operations undefined for certain arguments.

Let $\mathcal{M}_0^{\text{iaz}}$ be an inversive arithmetical meadow with zero. Then it makes sense to construct one partial inversive arithmetical meadow with zero from $\mathcal{M}_0^{\text{iaz}}$:

- $0^{-1} \upharpoonright \mathcal{M}_0^{\text{iaz}}$ is the partial algebra that is obtained from $\mathcal{M}_0^{\text{iaz}}$ by making 0^{-1} undefined.

Let $\mathcal{M}_0^{\text{daz}}$ be a divisive arithmetical meadow with zero. Then it makes sense to construct two partial divisive arithmetical meadows with zero from $\mathcal{M}_0^{\text{daz}}$:

- $Q / 0 \upharpoonright \mathcal{M}_0^{\text{daz}}$ is the partial algebra that is obtained from $\mathcal{M}_0^{\text{daz}}$ by making $q / 0$ undefined for all q in the domain of $\mathcal{M}_0^{\text{daz}}$;
- $(Q \setminus \{0\}) / 0 \upharpoonright \mathcal{M}_0^{\text{daz}}$ is the partial algebra that is obtained from $\mathcal{M}_0^{\text{daz}}$ by making $q / 0$ undefined for all q in the domain of $\mathcal{M}_0^{\text{daz}}$ different from 0.

Clearly, the partial arithmetical meadow constructions are special cases of a more general partial algebra construction for which we have coined the term *punching*. Presenting the details of the general construction is outside the scope of the current paper.

The partial arithmetical meadow constructions described above yield the following three partial arithmetical meadows with zero related to rational numbers:

$$0^{-1} \uparrow \mathcal{Q}_0^{\text{iaz}}, \quad \mathcal{Q} / 0 \uparrow \mathcal{Q}_0^{\text{daz}}, \quad (\mathcal{Q} \setminus \{0\}) / 0 \uparrow \mathcal{Q}_0^{\text{daz}}.$$

These algebras have been obtained by means of the well-known initial algebra construction and a simple partial algebra construction. The merits of this approach are discussed in [3].

At first sight, the absence of the additive inverse operator does not seem to add anything new to the treatment of punched meadows in [3]. However, this is not quite the case. Consider $0^{-1} \uparrow \mathcal{Q}_0^{\text{iaz}}$. In the case of this algebra, there is a useful syntactic criterion for “being defined”. The set *Def* of defined terms and the auxiliary set *Nz* of non-zero terms can be inductively defined by:

- $1 \in \text{Nz}$;
- if $x \in \text{Nz}$, then $x + y \in \text{Nz}$ and $y + x \in \text{Nz}$;
- if $x \in \text{Nz}$ and $y \in \text{Nz}$, then $x \cdot y \in \text{Nz}$;
- if $x \in \text{Nz}$, then $x^{-1} \in \text{Nz}$;
- $0 \in \text{Def}$;
- if $x \in \text{Nz}$, then $x \in \text{Def}$;
- if $x \in \text{Def}$ and $y \in \text{Def}$, then $x + y \in \text{Def}$ and $x \cdot y \in \text{Def}$.

This indicates that the absence of the additive inverse operator allows a typing based solution to problems related to “division by zero” in elementary school mathematics. So there may be a point in dealing first and thoroughly with non-negative rational numbers in a setting where division by zero is not defined.

Working in \mathcal{Q}^{ia} simplifies matters even more because there is no distinction between terms and defined terms. Again, this may be of use in the teaching of mathematics at elementary school.

8 Arithmetical Meadows and Regular Arithmetical Rings

We can define commutative arithmetical rings with a multiplicative identity in the same vein as arithmetical meadows. Moreover, we can define commutative von Neumann regular arithmetical rings with a multiplicative identity as commutative arithmetical rings with a multiplicative identity satisfying the regularity condition $\forall x \bullet \exists y \bullet x \cdot (x \cdot y) = x$.

The following theorem states that commutative von Neumann regular arithmetical rings with a multiplicative identity are related to inversive arithmetical meadows like commutative von Neumann regular rings with a multiplicative identity are related to inversive meadows.

Theorem 6. *Each commutative von Neumann regular arithmetical ring with a multiplicative identity can be expanded to an inversive arithmetical meadow, and this expansion is unique.*

Proof. Lemma 2.11 from [2] states that each commutative von Neumann regular ring with a multiplicative identity can be expanded to an inversive meadow, and this expansion is unique. The only use that is made of the equations $x + 0 = x$ and $x + (-x) = 0$ in the proof of this lemma given in [2] originates from the use of Lemma 2.12 from [2]. In the proof of the latter lemma, use is made of the equations $x + 0 = x$ and $x + (-x) = 0$. However, it is easy to see that the use of these equations can simply be avoided. By doing so, we obtain a proof of Lemma 2.11 from [2] that goes through for the arithmetical case. \square

We can also define commutative arithmetical rings with additive and multiplicative identities and commutative von Neumann regular arithmetical rings with additive and multiplicative identities in the obvious way. We also have that commutative von Neumann regular arithmetical rings with additive and multiplicative identities are related to inversive arithmetical meadows with zero like commutative von Neumann regular rings with a multiplicative identity are related to inversive meadows.

9 Conclusions

We have complemented the signatures of inversive and divisive meadows with arithmetical versions, and provided equational axiomatizations of several classes of arithmetical meadows and instances of them related to the field of rational numbers. We have answered a number of questions about these classes and instances of arithmetical meadows, and stated a number of outstanding questions about them. In addition, we have discussed partial variants of the instances in question and an arithmetical version of a well-known mathematical structure closely related to inversive meadows, namely von Neumann regular rings.

We remark that the name arithmetical algebra is not always used in the same way as Peacock [15] used it. It is sometimes difficult to establish whether the notion in question is related to Peacock's notion of arithmetical algebra. For example, it is not clear to us whether the notion of arithmetical algebra defined in [17] is related to Peacock's notion of arithmetical algebra.

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